Crypto-harmonic oscillator in higher dimensions: classical and quantum aspects

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Abstract

We study complexified harmonic oscillator models in two and three dimensions.
Our work is a generalization of the work of Smilga (2007 *Preprint* 0706.4064
(*J. Phys. A: Math. Theor.* at press)) who initiated the study of these Crypto-
gauge invariant models that can be related to PT-symmetric models. We show
that rotational symmetry in higher spatial dimensions naturally introduces more
constraints (in contrast to Smilga (2007 *Preprint* 0706.4064 (*J. Phys. A: Math.
Theor.* at press)) where one deals with a single constraint) with a much richer
constraint structure. Some common as well as distinct features in the study
of the same Crypto-oscillator in different dimensions are revealed. We also
quantize the two dimensional Crypto-oscillator.

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1. Introduction

It has been known for quite sometime [1] that there are quantum mechanical models with
specific complex terms in the Hamiltonian that admit real spectra and unitary evolution.
Later the seminal paper of Bender and Boettcher [2] attributed this intriguing and useful
property to the combined PT- (parity and time reversal) symmetry of the system and more
PT-symmetric models were constructed that had the above feature. Subsequently, there has
been a lot of activity [3] in the study of different aspects of PT-symmetric models. These
models are referred as 'Crypto'-Hermitian models by Smilga [4]. In [4] Smilga has also
provided an alternative explanation to this behavior (of having real energy eigenvalues for a
complex Hamiltonian): Crypto-gauge invariance. However, in an important earlier work by
Mostafazadeh [5], it was observed in a general context that the real part of the Hamiltonian can
generate the dynamics in a real phase space and that the imaginary part of the Hamiltonian,
treated as a constraint, can generate symmetry transformations. The usage of certain class of

coordinates in previous works [6] in related problems was also explained in [5]. The idea is to complexify a real Hamiltonian system and subsequently treat the real part of the complex $\mathcal{H}$ as the Hamiltonian H of the enlarged system with twice the original number of degrees of freedom. By virtue of Cauchy–Riemann condition (for $\mathcal{H}$) and Hamiltonian equations of motion it is possible to show that both the real part H and the imaginary part G of $\mathcal{H}$ (where $\mathcal{H} = H + \mathrm{i}G$) are *separately conserved*. This allows one to interpret G as a first class constraint (FCC) (see section 2 for a brief discussion on the constraint analysis as formulated by Dirac [7]) and in particular $G = 0$ ensures the reality of the energy value. This FCC is present in all such complexified systems and the gauge symmetry induced by this FCC [7] is termed as Crypto-gauge symmetry [4]. In [4], it has been shown that specific features of some complexified models (analyzed in terms of real variables) can be matched with their PT-symmetric counterpart in the complex plane.

It is important to emphasize that the work of Smilga [4] is restricted to one space dimension only and it naturally evokes the question of its application in higher dimensions. The present work specifically deals with this problem where we study the complexified or 'Crypto' harmonic oscillator (CHO) in two and three dimensions. The one-dimensional Crypto-oscillator was discussed by Smilga in [4]. As we will discuss at length in this paper, even this straightforward generalization reveals a number of interesting features that demand further study of higher dimensional Crypto-gauge systems in a model-independent way. It is worth mentioning that not much work has been done in PT-symmetric models in higher dimensions. Indeed, it will be very fruitful if our way of studying Crypto-gauge invariant models can reproduce results that are comparable with previously studied higher dimensional PT-symmetric models [8].

In this paper we will concentrate on additional spatial symmetries (such as rotational symmetries) that naturally occur in more than one dimensions. Following the same philosophy of demanding reality of energy values, which is a conserved quantity, one can also demand the reality of *other conserved quantities*, such as angular momentum, (as we have done here). This induces more constraints in the system and the subsequent analysis will require the Hamiltonian constraint analysis [7]. Our study will reveal a rich and interesting constraint structure for the higher dimensional models.

There seems to be still another way of interpreting the appearance of Crypto-gauge symmetry in this complexification process. In quantum field theories in the area of high energy physics there are several systematic procedures [9] of introducing gauge invariance (by way of FCCs) where the original model is embedded in a prescribed way in an extended phase space. The equivalence of the extended gauge invariant model with the original model is established in the so-called unitary gauge where the extended model reduces to the original one. Here, it is essential for the extended model to have the requisite number of FCCs that can account for the additional degrees of freedom in the enlarged phase space.

It is quite intriguing that the same phenomenon is repeated in the Crypto-gauge symmetric models although this was not quite apparent in the one-dimensional examples studied in [4]. In one dimension, complexification introduces one extra degree of freedom and there appears the FCC $G \approx 0$ to remove it. In fact, examples of unitary gauge choices have been given in [4]. On the other hand, in the higher dimensions that we consider (albeit in the CHO model) a larger number of degrees of freedom are introduced in the complexification process but quite surprisingly the number and nature of the additional constraints that appear from other conserved quantities (such as angular momentum) are just right to account for the extra variables. Indeed, it will be very interesting to establish this property in higher dimensions in a model-independent way.

2. Dirac constraint analysis—a brief digression

In the coordinate space formulation, starting from a Lagrangian $L(q_i, \dot{q}_i)$ of a dynamical system, constraints (if present) are revealed from the definition of the canonically conjugate momenta $p_i = (\partial L)/(\partial \dot{q}_i)$. In the Hamiltonian scheme [7], constraints are a set of relations $\varphi_a(q_i, p_i) \approx 0$, without any time derivative. The weak equality stresses the fact that the constraints can be put to zero $\varphi_a = 0$ as a strong equality only after all the relevant Poisson brackets are computed. Other constraints can appear from the requirement that the constraints are preserved in time and a complete set of constraints should obey

$$\{\varphi_a, H\} = 0 + \lambda_a \varphi_a.$$

Here, φ_a are a set of independent constraints and

$$H(p_i, q_i) = p_j \dot{q}_j - L + \lambda_a \varphi_a$$

is the canonical Hamiltonian modulo constraints. The Poisson bracket is computed by using the basic algebra

$$\{q_i, p_j\} = \delta_{ij}, \qquad \{q_i, q_j\} = \{p_i, p_j\} = 0.$$

Once the full set of constraints are obtained Dirac introduced the very important classification of constraints. If in the full set φ_a there are constraints F_α that (Poisson) commute with *all* the constraints,

$$\{F_\alpha, \varphi_a\} = 0 + \lambda_{\alpha ab} \varphi_b,$$

the set F_α is termed as first class constraints (FCC). The rest of the constraints H_β that do not commute with all the constraints are termed as second class constraints (SCC). In practical terms this means that the constraint matrix, with $\{\varphi_a, \varphi_b\}$ as matrix elements, will be *degenerate* if there are FCCs in the system and it will be invertible if only SCCs are present.

The FCCs are responsible for local gauge invariances in the system and they are related to the generators of local gauge transformations. On the other hand, the SCCs induce a modification in the symplectic structure and one has to replace the basic Poisson Brackets by a new set of brackets, known as Dirac brackets. Also it is important to point out that the presence of FCCs indicate that there are redundant variables that are not physical degrees of freedom and one is allowed to choose additional constraints, known as gauge fixing conditions, that can remove these trivial variables. Note that a system of FCCs together with proper gauge fixing constraints becomes a set of SCCs.

An SCC can be used to eliminate one degree of freedom in phase space. On the other hand, one FCC, together with an associated gauge fixing constraint, constitutes a pair of SCCs and accounts for two degrees of freedom in phase space. In this way, one can determine the true degrees of freedom of a constrained system.

The idea is that in quantizing a system with second class constraints one needs to elevate the Dirac brackets (and *not* the Poisson brackets) to quantum commutators.

In the present work we will only invoke the idea of the classification of constraints and explicit construction of the Dirac brackets will be left for a future publication.

3. Two-dimensional CHO: classical analysis

The CHO Hamiltonian is

$$\mathcal{H}(\pi_i, z_i) = \frac{\pi_i{}^2 + z_i{}^2}{2} \tag{1}$$

where $i = 1, 2$. Clearly, this is just the two-dimensional extension of the construction of Smilga [4]. Following [4] we express the complex phase space variables $(\pi_i; z_i)$ in terms of real phase space variables $z_i = x_i + iy_i$, $\pi_i = p_i - iq_i$. The above phase space is canonical with the only non-vanishing Poisson brackets being $\{x_i, p_j\} = \delta_{ij}$; $\{y_i, q_j\} = \delta_{ij}$. The complex Hamiltonian $\mathcal{H}$ in (1) now reads

$$\mathcal{H}(\pi_i, z_i) = H(p_i, q_i; x_i, y_i) + iG(p_i, q_i; x_i, y_i),$$
$$H = \tfrac{1}{2}\big[\big(p_i{}^2 + x_i{}^2\big) - \big(q_i{}^2 + y_i{}^2\big)\big], \qquad G = -p_i q_i + x_i y_i. \tag{2}$$

In order to restrict the classical Hamiltonian to the real space, we impose the constraint $G \approx 0$ where the weak equality is interpreted in the sense of Dirac [7]. As noted in [4], G (Poisson) commutes with H: $\{G, H\} = 0$ that can be checked explicitly. So far everything appears to be a straightforward extension of [4] but now comes the new elements.

In two dimensions one can moot the idea of a complex angular momentum and demand its reality. The complex angular momentum is defined as

$$\mathcal{L} = z_1 \pi_2 - z_2 \pi_1 \equiv L_R + iL_G, \tag{3}$$

$$L_R = \epsilon_{ij}(x_i p_j + y_i q_j), \qquad L_G = -\epsilon_{ij}(x_i q_j - y_i p_j). \tag{4}$$

For real values of angular momentum we impose $L_G \approx 0$. The angular momentum L_R is a conserved quantity $\{L_R, H\} = 0$.

The two-dimensional CHO has two constraints $G \approx 0$, $L_G \approx 0$ ([4] had one) and so we will require a full constraint analysis [7], as discussed in section 2. First of all one has to obtain the full set of linearly independent constraints such that the constraint system is stable under time translation. In the present case this is ensured by noting

$$\{G, H\} \approx 0, \qquad \{L_G, H\} \approx 0. \tag{5}$$

Next comes the classification of the constraints. In our system,

$$\{L_G, G\} \approx 0. \tag{6}$$

This shows that both the constraints are FCC in nature (of the type F_α mentioned in section 2).

There are two generic features that are common in the one-dimensional model [4] and its higher dimensional extensions studied here.

First one is the fact that the constraint that is generated from the reality of angular momentum commutes with H. This property remains valid in the three-dimensional extension as well and this type of additional constraints did not appear in one-dimensional case [4]. This property might be a particular feature of the CHO model. Remember that for the constraint G that originated from the complex Hamiltonian, one can exploit the Cauchy–Riemann conditions to show $\{G, H\} = 0$ in a model-independent way. It will be interesting to see if our result has a deeper significance.

The second point is related to the degrees of freedom count. Note that in [4] in one dimension one extra degree of freedom was introduced due to complexification and it can be removed by the single FCC G. This is because the additional two variables (y, q) in phase space can be removed by the FCC G and a suitable gauge choice (the so-called unitary gauge). Now in two dimensions, the extension is by two degree of freedom (four variables $(y_i, q_i; i = 1, 2)$ in phase space but now there are two FCCs G and L_G (along with two gauge choices) to account for them). Hence effectively the number of degrees of freedom has not changed in the process of complexification. This property is preserved in three dimensions as well but in a more interesting and non-trivial way.

A constrained Lagrangian for the CHO is

$$L = x_i \dot{p}_i + y_i \dot{q}_i - H + \lambda_1 G + \lambda_2 L_G, \tag{7}$$

with λ_1, λ_2 being Lagrange multipliers. From the Euler–Lagrange equations of motion we obtain

$$p_i = \dot{x}_i - \lambda_1 q_i - \lambda_2 \epsilon_{ij} y_j, \qquad q_i = -\dot{y}_i + \lambda_1 p_i - \lambda_2 \epsilon_{ij} x_j. \tag{8}$$

Substituting the momenta into (7) and finally eliminating the multipliers λ_1, λ_2 we can get the coordinate space Lagrangian. One can check that it is invariant under the gauge transformations generated by G and L_G.

In the present work we will not try to develop the full dynamics of the model but will only show that the model admits closed trajectories for positive energies and angular momentum, in a partially gauge fixed setup (similar to [4]) with $\lambda_1 = \lambda_2 = 0$. Let us consider the simplest possible bounded solution,

$$x_i = A_i \cos(t) + B_i \sin(t); \qquad y_i = R_i \cos(t) + Q_i \sin(t), \tag{9}$$

where A_i, B_i, R_i, Q_i are time independent parameters. Substituting (9) into the previously computed expressions for the Hamiltonian H, angular momentum L_R and constraints G, L_G, we obtain

$$H = \tfrac{1}{2}[(A^2 + B^2) - (R^2 + Q^2)]; \qquad G = AR + BQ$$
$$L_R = \epsilon_{ij}(A_i B_j + Q_i R_j); \qquad L_G = \epsilon_{ij}(A_i Q_j - B_i R_j). \tag{10}$$

Now consider the following choices of A, B, Q, R for which both the constraints vanish and H and L_R take different forms:

(i) $A_i = -\epsilon_{ij} B_j$; $Q_i = -\epsilon_{ij} R_j \Rightarrow H \equiv E = A^2 - R^2, L_R = -(A^2 + R^2)$,
(ii) $A_i = B_i = 0 \Rightarrow E = -\tfrac{1}{2}(R^2 + Q^2), L_R = \epsilon_{ij} Q_i R_j$,
(iii) $A_i = \pm\epsilon_{ij} R_j$; $B_i = \pm\epsilon_{ij} Q_j \Rightarrow E = L_R = 0$,
(iv) $A_i = \epsilon_{ij} B_j$; $Q_i = \epsilon_{ij} R_j \Rightarrow E = A^2 - R^2, L_R = A^2 + R^2$,
(v) $R_i = Q_i = 0 \Rightarrow E = \tfrac{1}{2}(A^2 + B^2), L_R = \epsilon_{ij} Q_i R_j$.

Let us now comment on these alternative possibilities: clearly the choices (i) and (ii) are not interesting because for classical systems, energy or angular momentum can not be negative. Also (iii) does not represent a dynamical system since both energy and angular momentum vanish. The choices (iv) and (v) are physically relevant. Obviously, (v) represents the conventional harmonic oscillator. Let us focus our attention on (iv). Here the energy E is not positive definite but angular momentum L_R is positive definite. Turning this around, we might demand both positive definite values for E and L_R and in that case we can plot the constant (positive) energy and angular surfaces to get an idea of the particle trajectory. Clearly, a fixed positive energy can give rise to unbounded motion in the form of open surfaces whereas a fixed angular momentum will lead to a closed surface (in fact a hyper-sphere). Hence their intersection will yield a closed trajectory. This is shown in the figure where both the surfaces are plotted with the coordinate $y_2 = 0$.

4. Three-dimensional CHO: classical analysis

The three-dimensional CHO is studied in the same way as before where equations (1), (2) for the complex Hamiltonian remain structurally identical with $i = 1, 2, 3$. $\mathcal{H}$ will be real provided $G \approx 0$ is treated as a constraint.

Proceeding in the same way as we did in the two-dimensional counterpart in section 3 we define the ith component of the angular momentum as

$$L^i = L_R^i + i L_G^i; \qquad L_R^i = \epsilon^{ijk}(x^j p^k + y^j q^k), \qquad L_G^i = \epsilon^{ijk}(y^j p^k - x^j q^k). \qquad (11)$$

We impose further constraints $L_G^i \approx 0$ to keep the reality of angular momenta intact. From $\{L_R^i, H\} = 0, \{L_R^i, L_R^j\} = \epsilon^{ijk} L_R^k$ we find L_R is conserved and preserve $SO(3)$ algebra.

Next we carry out the constraint analysis with the four constraints, $G \approx 0, L_G^i \approx 0, i = 1, 2, 3$. From $\{G, H\} = 0, \{L_G^i, H\} = 0$, we find that the system of constraints is stable against time translation. Next the constraint algebra $\{L_G^i, G\} = 0, \{L_G^i, L_G^j\} = -\epsilon^{ijk} L_R^k$ indicates that G is an FCC (of the type F_α) but also there are SCCs (of the type H_β). Since there cannot be an odd number of SCCs[3] (three in the present case) there has to be another FCC. Taking help from the rest of the algebra $\{L_R^i, G\} = 0, \{L_G^i, L_R^j\} = \epsilon^{ijk} L_G^k$ we find that the following combination, $W \equiv L_R^i L_G^i \approx 0$, constitutes the other FCC. Hence, we conclude that the system has two FCCs $G \approx 0, W \approx 0$ (of type F_α in section 2) and two SCCs which we can be chosen as L_G^1, L_G^2 (of type H_β of section 2) with the non-vanishing bracket

$$\{L_G^1, L_G^2\} = -L_R^3. \qquad (12)$$

Let us consider the degrees of freedom count in the presence of the constraints. In three dimensions we have introduced three additional degrees of freedom and they can be accounted for by the two FCCs (each removing one degree of freedom) and the pair of SCC (the latter together removes one degree of freedom). In this sense, the parity is once again restored between the number of degrees of freedom in the original system and the constrained 'Crypto' system.

Although we will not pursue the quantization of the three-dimensional CHO in the present work we note that the closed algebra of L_R^i, L_G^j is nothing but the group algebra of $SL(2, C)$. We also stress that in the oscillator basis were L_R^3 is diagonal, the SCC structure $\{L_G^1, L_G^2\} = -L_R^3$ is *not operator valued* and so the quantization should not be problematic. Interestingly, for the zero angular momentum state $\{L_G^1, L_G^2\} = 0$ meaning that there are no SCC for this particular state. But even with four FCCs the degrees of freedom still matches because remember that the zero angular momentum state will depend only on the planar distance and not on the angle.

5. Two-dimensional CHO: quantum analysis

In this section we discuss the quantization of the planar CHO. Following the procedure one adopts in the case of a normal HO, we define two sets of lowering operators as

$$a_i = \tfrac{1}{\sqrt{2}}(p_i - i x_i), \qquad b_i = \tfrac{1}{\sqrt{2}}(q_i - i y_i), \qquad (13)$$

with the non-zero commutator, $[a_i, a_j^\dagger] = [b_i, b_j^\dagger] = \delta_{ij}$. Next we define the Schwinger operators

$$A_1 = \tfrac{1}{\sqrt{2}}(a_1 + i a_2); \qquad A_2 = \tfrac{1}{\sqrt{2}}(a_1 - i a_2)$$
$$B_1 = \tfrac{1}{\sqrt{2}}(b_1 + i b_2); \qquad B_2 = \tfrac{1}{\sqrt{2}}(b_1 - i b_2).$$

The only non-zero commutators are $[A_i, A_j^\dagger] = [B_i, B_j^\dagger] = \delta_{ij}$. The advantage of using A_i, B_i is that both the Hamiltonian H as well as the single component of angular momentum L_R are diagonal when expressed in terms of A_i, B_i. Hence we find

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[3] Remember that the constraint matrix for SCCs is non-singular.

$$H = N_{A_1} + N_{A_2} - N_{B_1} - N_{B_2}, \qquad L_R = N_{A_2} + N_{B_2} - N_{A_1} - N_{B_1}, \tag{15}$$

$$G = -\left(A_1 B_2 + A_2 B_1 + A_1^\dagger B_2^\dagger + A_2^\dagger B_1^\dagger\right), \qquad L_G = A_1 B_2 - A_2 B_1 + A_1^\dagger B_2^\dagger - A_2^\dagger B_1^\dagger \tag{16}$$

where the number operators are defined as $N_{A_1} = A_1^\dagger A_1$, etc. Since H and L_R commute, it is possible to choose a common eigen-basis of both H and L_R. We choose the common eigen-basis as $|n_{A_1}, n_{B_1}; n_{A_2}, n_{B_2}\rangle$ with the following action of the Schwinger operators on them:

$$A_1 |n_{A_1}, n_{B_1}; n_{A_2}, n_{B_2}\rangle = \sqrt{n_{A_1}} |n_{A_1} - 1, n_{B_1}; n_{A_2}, n_{B_2}\rangle$$
$$A_1^\dagger |n_{A_1}, n_{B_1}; n_{A_2}, n_{B_2}\rangle = \sqrt{n_{A_1} + 1} |n_{A_1} + 1, n_{B_1}; n_{A_2}, n_{B_2}\rangle. \tag{17}$$

The actions of the rest of the operators A_2, $A_2^\dagger$, B_1, B_2, $B_1^\dagger$, $B_2^\dagger$ are similar. Eigenvalues for H and L_R are given as

$$H |n_{A_1}, n_{B_1}; n_{A_2}, n_{B_2}\rangle = \left(n_{A_1} + n_{A_2} - n_{B_1} - n_{B_2}\right) |n_{A_1}, n_{B_1}; n_{A_2}, n_{B_2}\rangle$$
$$L_R |n_{A_1}, n_{B_1}; n_{A_2}, n_{B_2}\rangle = \left(n_{A_2} + n_{B_2} - n_{A_1} - n_{B_1}\right) |n_{A_1}, n_{B_1}; n_{A_2}, n_{B_2}\rangle. \tag{18}$$

Any state can be written as a linear combination in the above basis,

$$|\Psi\rangle = \sum_{n_{A_1}, \ldots, n_{B_2} = 0}^{\infty} C_{n_{A_1}, \ldots, n_{B_2}} |n_{A_1}, n_{B_1}; n_{A_2}, n_{B_2}\rangle. \tag{19}$$

Now comes the role of the constraints. Since they are FCCs we follow the Dirac formalism [7] and pick the physical sector by demanding that the FCCs kill the physical states $(FCC)|\Psi^{ph}\rangle = 0$ which in the present case means

$$G|\Psi^{ph}\rangle = 0; \qquad L_G|\Psi^{ph}\rangle = 0. \tag{20}$$

However, in the present problem, it is more convenient to impose the linear combinations of FCCs,

$$(G + L_G)|\Psi^{ph}\rangle = 0; \qquad (G - L_G)|\Psi^{ph}\rangle = 0. \tag{21}$$

Considering the first one $(G + L_G)|\Psi^{ph}\rangle = 0$, we find

$$\sum_{n_{A_1}, \ldots, n_{B_2} = 0}^{\infty} C_{n_{A_1}, \ldots, n_{B_2}} \left[\sqrt{n_{A_2} n_{B_1}} |n_{A_1}, n_{B_1} - 1; n_{A_2} - 1, n_{B_2}\rangle \right.$$

$$\left. + \sqrt{(n_{A_2} + 1)(n_{B_1} + 1)} |n_{A_1}, n_{B_1} + 1; n_{A_2} + 1, n_{B_2}\rangle \right] = 0. \tag{22}$$

To find the states that satisfy (22) with arbitrary energy m (including zero) and arbitrary values of angular momentum n (including zero), we use (18) and obtain the conditions

$$n_{A_1} + n_{A_2} = m + n_{B_1} + n_{B_2}, \qquad n_{A_2} - n_{A_1} = n + n_{B_1} - n_{B_2}, \tag{23}$$

where $m = -\infty$ to $+\infty$ and $n = 0, 1, 2, 3, \ldots$. The numbers n_A, n_B are the eigenvalues of the corresponding number operators N_A, N_B, etc. Solving the above two equations we get

$$n_{B_1} = n_{A_2} - \frac{m}{2} - \frac{n}{2}, \qquad n_{B_2} = n_{A_1} - \frac{m}{2} + \frac{n}{2}. \tag{24}$$

Substituting these into (22) we have

$$\sum_{n_{A_1}, n_{A_2}=0}^{\infty} C_{n_{A_1}, n_{A_2}} \left[\sqrt{\left(n_{A_2} - \frac{m}{2} - \frac{n}{2}\right) n_{A_2}} \left| n_{A_1}, n_{A_2} - \frac{m}{2} - \frac{n}{2} - 1; n_{A_2} - 1, n_{A_1} + \frac{n}{2} - \frac{m}{2} \right\rangle \right.$$

$$+ \sqrt{(n_{A_2} + 1)\left(n_{A_2} - \frac{m}{2} - \frac{n}{2} + 1\right)} \left| n_{A_1}, n_{A_2} - \frac{m}{2} - \frac{n}{2} + 1; \right.$$

$$\left. \left. n_{A_2} + 1, n_{A_1} + \frac{n}{2} - \frac{m}{2} \right\rangle \right]$$

$$= 0 \tag{25}$$

Replacing n_{A_1} by n_1 and n_{A_2} by n_2 and then substituting n_2 by $n_2 - 2$ into the second term of the above relation we get

$$\sum_{n_1, n_2=0}^{\infty} C_{n_1, n_2} \sqrt{\left(n_2 - \frac{m}{2} - \frac{n}{2}\right) n_2} \left| n_1, n_2 - \frac{m}{2} - \frac{n}{2} - 1; n_2 - 1, n_1 - \frac{m}{2} + \frac{n}{2} \right\rangle$$

$$+ \sum_{n_1=0, n_2=2}^{\infty} C_{n_1, n_2-2} \sqrt{\left(n_2 - \frac{m}{2} - \frac{n}{2} - 1\right)(n_2 - 1)} \left| n_1, n_2 - \frac{m}{2} - \frac{n}{2} - 1; \right.$$

$$\left. \left. n_2 - 1, n_1 - \frac{m}{2} + \frac{n}{2} \right\rangle \right]$$

$$= 0. \tag{26}$$

Putting $C_{n_1,-2} = C_{n_1,-1} = 0$ we can rewrite the above as

$$\sum_{n_1, n_2=0}^{\infty} \left[C_{n_1, n_2} \sqrt{\left(n_2 - \frac{m}{2} - \frac{n}{2}\right) n_2} + C_{n_1, n_2-2} \sqrt{\left(n_2 - \frac{m}{2} - \frac{n}{2} - 1\right)(n_2 - 1)} \right]$$

$$\left| n_1, n_2 - \frac{m}{2} - \frac{n}{2} - 1; n_2 - 1, n_1 - \frac{m}{2} + \frac{n}{2} \right\rangle$$

$$= 0. \tag{27}$$

Since the basis vectors are linearly independent, the coefficient within the third bracket must vanish for each basis vectors and hence we have the following recursion relation:

$$C_{n_1, n_2} = -\sqrt{\frac{\left(n_2 - \frac{m}{2} - \frac{n}{2} - 1\right)(n_2 - 1)}{\left(n_2 - \frac{m}{2} - \frac{n}{2}\right) n_2}} C_{n_1, n_2-2} \tag{28}$$

where $n_1 = 0, 1, 2, 3, \dots$ and $n_2 = 2, 3, 4, \dots$ since $C_{n_1,-2} = C_{n_1,-1} = 0$. Using the above recursion relation one can show

$$C_{i, 2k} = (-1)^k \sqrt{\frac{\left(2k - 1 - \frac{m}{2} - \frac{n}{2}\right)!! (2k - 1)!!}{\left(2k - \frac{m}{2} - \frac{n}{2}\right)!! (2k)!!}} C_{i,0};$$

$$C_{i, 2k+1} = (-1)^k \sqrt{\frac{\left(2k - \frac{m}{2} - \frac{n}{2}\right)!! (2k)!!}{\left(2k + 1 - \frac{m}{2} - \frac{n}{2}\right)!! (2k + 1)!!}} C_{i,1} \tag{29}$$

where $i = 0, 1, 2, \ldots, k = 1, 2, 3, \ldots$ and

$$
(2k)!! = 2.4.6 \ldots (2k-2)(2k);
$$

$$
(2k-1)!! = 1.3.5 \ldots (2k-3)(2k-1);
$$

$$
\left(2k-1-\frac{m}{2}-\frac{n}{2}\right)!! = \left(1-\frac{m}{2}-\frac{n}{2}\right)\left(3-\frac{m}{2}-\frac{n}{2}\right)\ldots \tag{30}
$$

$$
\left(2k-3-\frac{m}{2}-\frac{n}{2}\right)\left(2k-1-\frac{m}{2}-\frac{n}{2}\right).
$$

Therefore, by imposing one of the FCCs we narrow down the physical sector to the following state:

$$
\left|\Psi_{m,n}^{ph}\right\rangle = \sum_{i=0,k=1}^{\infty} (-1)^k \left[\sqrt{\frac{\left(2k-1-\frac{m}{2}-\frac{n}{2}\right)!!(2k-1)!!}{\left(2k-\frac{m}{2}-\frac{n}{2}\right)!!(2k)!!}} C_{i,0} \left| i, 2k-\frac{m}{2}-\frac{n}{2}; 2k, i-\frac{m}{2}+\frac{n}{2}\right\rangle \right.
$$

$$
\left. + \sqrt{\frac{\left(2k-\frac{m}{2}-\frac{n}{2}\right)!!(2k)!!}{\left(2k+1-\frac{m}{2}-\frac{n}{2}\right)!!(2k+1)!!}} C_{i,1} \left| i, 2k+1-\frac{m}{2}-\frac{n}{2}; 2k+1, i-\frac{m}{2}+\frac{n}{2}\right\rangle \right]. \tag{31}
$$

Finally, we further restrict the sector to the correct physical one by imposing the other FCC,

$$
(G - L_G)\left|\Psi_{m,n}^{ph}\right\rangle = -2\left(A_1 B_2 + A_1^{\dagger} B_2^{\dagger}\right)\left|\Psi_{m,n}^{ph}\right\rangle = 0. \tag{32}
$$

Another relation between the parameters follows:

$$
\sum_{i=0,k=1}^{\infty} (-1)^k \left[\sqrt{\frac{\left(2k-1-\frac{m}{2}-\frac{n}{2}\right)!!(2k-1)!!}{\left(2k-\frac{m}{2}-\frac{n}{2}\right)!!(2k)!!}} \right.
$$

$$
\times C_{i,0} \left\{ \sqrt{i\left(i-\frac{m}{2}+\frac{n}{2}\right)} \left| i-1, 2k-\frac{m}{2}-\frac{n}{2}; 2k, i-\frac{m}{2}+\frac{n}{2}-1\right\rangle \right.
$$

$$
\left. + \sqrt{(i+1)\left(i+1-\frac{m}{2}+\frac{n}{2}\right)} \left| i+1, 2k-\frac{m}{2}-\frac{n}{2}; 2k, i-\frac{m}{2}+\frac{n}{2}+1\right\rangle \right\}
$$

$$
+ \sqrt{\frac{\left(2k-\frac{m}{2}-\frac{n}{2}\right)!!(2k)!!}{\left(2k+1-\frac{m}{2}-\frac{n}{2}\right)!!(2k+1)!!}}
$$

$$
\times C_{i,1} \left\{ \sqrt{i\left(i-\frac{m}{2}+\frac{n}{2}\right)} \left| i-1, 2k+1-\frac{m}{2}-\frac{n}{2}; 2k+1, i-\frac{m}{2}+\frac{n}{2}-1\right\rangle \right.
$$

$$
\left.\left. + \sqrt{(i+1)\left(i-\frac{m}{2}+\frac{n}{2}+1\right)} \left| i+1, 2k+1-\frac{m}{2}-\frac{n}{2}; 2k+1, i-\frac{m}{2}+\frac{n}{2}+1\right\rangle \right\} \right]
$$

$$
= 0. \tag{33}
$$

Explicitly writing the above equation for the sum over $i = 0$ to ∞ one can show

$$
C_{\mu\nu} = 0,
$$

$$
C_{2r,0} = (-1)^r \sqrt{\frac{(2r-1)!!\left(2r-1-\frac{m}{2}+\frac{n}{2}\right)!!}{(2r)!!\left(2r-\frac{m}{2}+\frac{n}{2}\right)!!}} C_{0,0},
$$

$$
C_{2r,1} = (-1)^r \sqrt{\frac{(2r-1)!!\left(2r-1-\frac{m}{2}+\frac{n}{2}\right)!!}{(2r)!!\left(2r-\frac{m}{2}+\frac{n}{2}\right)!!}} C_{0,1}, \tag{34}
$$

where $\mu = 1, 3, 5, 7, \ldots$, $\nu = 0, 1$ and $r = 1, 2, 3, 4, \ldots$. So, the final form of the physical state for arbitrary energy m and angular momentum n is

$$
\begin{aligned}
\left|\Psi_{m,n}^{ph}\right\rangle = \sum_{r,k=1}^{\infty} (-1)^{k+r} &\sqrt{\frac{(2r-1)!!\left(2r-1-\frac{m}{2}+\frac{n}{2}\right)!!}{(2r)!!\left(2r-\frac{m}{2}+\frac{n}{2}\right)!!}} \\
&\times \left[\sqrt{\frac{\left(2k-1-\frac{m}{2}-\frac{n}{2}\right)!!(2k-1)!!}{\left(2k-\frac{m}{2}-\frac{n}{2}\right)!!(2k)!!}}\, C_{0,0}\left|2r, 2k-\frac{m}{2}-\frac{n}{2}; 2k, 2r-\frac{m}{2}+\frac{n}{2}\right\rangle \right. \\
&\left. + \sqrt{\frac{\left(2k-\frac{m}{2}-\frac{n}{2}\right)!!(2k)!!}{\left(2k+1-\frac{m}{2}-\frac{n}{2}\right)!!(2k+1)!!}}\, C_{0,1} \right. \\
&\left. \times \left|2r, 2k+1-\frac{m}{2}-\frac{n}{2}; 2k+1, 2r-\frac{m}{2}+\frac{n}{2}\right\rangle\right].
\end{aligned}
\tag{35}
$$

With this we conclude the quantization of the two-dimensional Crypto-oscillator.

It is also straightforward to recover the quantum version of one-dimensional CHO that was discussed in [4]. In one dimension, x_2, p_2, y_2, q_2 are absent from the set (13) which means that in (14) $A_1 = A_2 \equiv A$, $B_1 = B_2 \equiv B$. Putting this back into (16,16), we obtain

$$
H = 2(N_A - N_B), \qquad G = -2(AB + A^\dagger B^\dagger), \qquad L_R = 0, \qquad L_G = 0, \tag{36}
$$

which is nothing but the model studied in [4].

6. Summary and outlook

In this paper we have generalized the Crypto harmonic oscillator model, proposed by Smilga [4], to higher (two and three) dimensions. After complexification, the energy is restricted to the real sector by demanding that the imaginary part of the energy vanishes. This introduces a (Hamiltonian) constraint in the theory [4]. In higher dimensions there are other physical dynamical variables (such as angular momentum that is considered here) besides the energy and it is only natural to restrict them to the real sector as well. This brings in additional constraints and a formal constraint analysis [7] reveals interesting features. Also we have quantized the two-dimensional Crypto harmonic oscillator in the present paper.

An interesting problem is to ascertain to what extent the new features in the constraint structure revealed here in the higher dimensional extension are model independent. If these features turn out to be generic, then this formalism can be still another alternative way of introducing gauge symmetry via phase space extension. In fact, we are now studying the Crypto version of the oscillator with a position-dependent effective mass and there also these features persist. These results will be reported elsewhere.

The other problem is obviously to apply this idea of Crypto-gauge invariance, as adapted in our work in higher space dimensions, to more complicated models and to compare the results with the analog higher dimensional PT-symmetric models.

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